Topology and its Applications 158 (2011) 930–938

Contents lists available at ScienceDirect

Topology and its Applications

www.elsevier.com/locate/topol

Permutable pairs of quasi-uniformities

Eliza P. de Jager, Hans-Peter A. Künzi *,1

Department of Mathematics and Applied Mathematics, University of Cape Town, Rondebosch 7701, South Africa

A R T I C L E   I N F O

Article history:
Dedicated to Professor Dikran Dikranjan on the occasion of his sixtieth birthday

MSC:
54E15
54E55
54D15
54G05
06B05

Keywords:
Lattice of quasi-uniformities
Permutable quasi-uniformities
Complementary quasi-uniformities
Pervin quasi-uniformity
Almost 2-fully normal

A B S T R A C T

We continue investigating the lattice $(q(X), \subseteq)$ of quasi-uniformities on a set X. In particular in this article we start investigating permutable pairs of quasi-uniformities. Among other things, we show that the Pervin quasi-uniformity of a topological space X permutes with its conjugate if and only if X is normal and extremally disconnected.

1. Introduction

It is well known that the set $q(X)$ of all quasi-uniformities on a given set X yields a complete lattice provided that it is partially ordered under set-theoretic inclusion $\subseteq$. That lattice $(q(X), \subseteq)$ was studied in [4–6,8]. Some open questions in the area were discussed in [7].

In the present article we embark on first investigations about permutable pairs of quasi-uniformities. Various deep results about permutable families of uniformities were recently obtained by Weber [25,26].

In the present article we prove among other things that a topological space X is normal and extremally disconnected if and only if the Pervin quasi-uniformity $\mathcal{P}$ of X and its conjugate $\mathcal{P}^{-1}$ permute.

2. Preliminary remarks

Let us first recall some definitions. We shall call a reflexive transitive (binary) relation on a set X a *preorder*. As usual, a preorder that is antisymmetric will be called a *partial order*. For binary relations A and B on a set X we set $B \circ A = \{(x, z) \in X \times X\colon$ there is $y \in X$ such that $(x, y) \in A$ and $(y, z) \in B\}$. For a subset C of a topological space X, $\bar{C}$ will denote the *closure* of C and $\operatorname{int} C$ the *interior* of C in X.

A filter $\mathcal{U}$ on $X \times X$ such that each $U \in \mathcal{U}$ is a reflexive relation and for each $U \in \mathcal{U}$ there is $V \in \mathcal{U}$ such that $V \circ V \subseteq U$ is called a *quasi-uniformity* on X. For basic facts about quasi-uniformities we refer the reader to [11,16]. In recent years

* Corresponding author.
E-mail addresses: epdejager@yahoo.com (E.P. de Jager), hans-peter.kunzi@uct.ac.za (H.-P.A. Künzi).

1 The author would like to thank the South African National Research Foundation for partial financial support.

0166-8641/$ – see front matter
doi:10.1016/j.topol.2011.01.012

studies on quasi-uniform spaces mainly concentrated on hyperspaces, asymmetric functional analysis, pointfree topology and applications of asymmetric topology to computer science (see e.g. [1,10,21,22]).

We finish this section by citing some further concepts and results from the theory of quasi-uniform spaces that we shall use throughout this article. Note that for any quasi-uniformity $\mathcal{U}$ the filter $\mathcal{U}^{-1} = \{U^{-1}\colon U \in \mathcal{U}\}$, where $U^{-1} = \{(y,x) \in X \times X\colon (x,y) \in U\}$ denotes the relation inverse to U, is also a quasi-uniformity on X. A quasi-uniformity $\mathcal{U}$ satisfying $\mathcal{U} = \mathcal{U}^{-1}$ is called a *uniformity*. The topology $\tau(\mathcal{U})$ *induced by* $\mathcal{U}$ *on* X consists of all subsets G of X such that for each $x \in G$ there is $U \in \mathcal{U}$ such that $U(x) \subseteq G$ where $U(x) = \{y \in X\colon (x,y) \in U\}$. A quasi-uniformity is called *transitive* provided that it has a base of transitive relations [11, p. 27].

The smallest element of the lattice $(q(X), \subseteq)$ is the indiscrete uniformity $\mathcal{I} = \{X \times X\}$, while the largest element of $(q(X), \subseteq)$ is the discrete uniformity $\mathcal{D}$ generated by the base $\{\Delta\}$, where $\Delta = \{(x,x)\colon x \in X\}$ is the *diagonal of* X. The lattice of preorders on a set X embeds as a sublattice (see [4, p. 3153]) into the lattice of quasi-uniformities $(q(X), \subseteq)$ on X, via the embedding $T \mapsto \mathcal{U}_T$ where for each preorder T, $\mathcal{U}_T$ is the quasi-uniformity on X having the base $\{T\}$: Indeed if T_1 and T_2 are preorders on a set X, then $\mathcal{U}_{T_1} \vee \mathcal{U}_{T_2}$ is equal to $\mathcal{U}_{T_1 \cap T_2}$, and if $(T_i)_{i \in I}$ is any nonempty family of preorders on X, then $\bigwedge_{i \in I} \mathcal{U}_{T_i}$ is equal to $\mathcal{U}_S$ where S is the preorder $\bigcup_{n \in \mathbb{N}} (\bigcup_{i \in I} T_i)^n$ (see [4, p. 3154]; the proof given there for the case of two preorders works in general; compare with [25, p. 255]).

Two quasi-uniformities $\mathcal{U}$ and $\mathcal{V}$ on a set X are called *complementary* if $\mathcal{U} \vee \mathcal{V} = \mathcal{D}$ and $\mathcal{U} \wedge \mathcal{V} = \mathcal{I}$, where $\vee$ and $\wedge$ denote the lattice operations of $(q(X), \subseteq)$.

For any subset A of X we set $S_A = [(X \setminus A) \times X] \cup [X \times A]$; furthermore let $C_A = \Delta \cup [A \times (X \setminus A)]$. Then for any subset A of X, S_A is a preorder and C_A is a partial order on X.

It is known that a quasi-uniformity $\mathcal{U}$ on a set X is an atom in $(q(X), \subseteq)$ if and only if $\mathcal{U}$ is of the form $\mathcal{U}_{S_A}$ where A is a nonempty proper subset of X [4, Propositions 1 and 2].

The compatible *Pervin quasi-uniformity* of a topological space X is generated by the subbase $\{S_G\colon G \text{ is open in } X\}$ (see [11, p. 28]).

For any subset A of X the quasi-uniformities $\mathcal{U}_{S_A}$ and $\mathcal{U}_{C_A}$ are complementary in $(q(X), \subseteq)$, since $S_A \cap C_A = \Delta$ and $S_A \cup C_A = X \times X$ (compare with [8, Remark 1]).

3. Permutable pairs of quasi-uniformities

The theory of permuting (= commuting) pairs of equivalence relations is highly developed (compare e.g. [27]). In [25,26] Weber extended some of the results of this theory to permuting families of uniformities. On the other hand, very little seems to be known about the asymmetric case. In [28] Yan investigated pairs of permuting preorders. In the following we want to explore this idea in the spirit of Weber by studying more generally pairs of permuting quasi-uniformities.

Given two quasi-uniformities $\mathcal{U}$ and $\mathcal{V}$ on a set X, we let $\mathcal{U} \circ \mathcal{V}$ be the filter on $X \times X$ generated by the base $\{U \circ V\colon U \in \mathcal{U},\ V \in \mathcal{V}\}$. We shall say that $\mathcal{U} \circ \mathcal{V}$ is *symmetric* provided that it is equal to the filter $(\mathcal{U} \circ \mathcal{V})^{-1}$.

Definition 1. (Compare [25, Definition 3.1] or [23, Section 6.1].) Two quasi-uniformities $\mathcal{U}$ and $\mathcal{V}$ on a set X are called *permutable* (or said to *permute*) if $\mathcal{U} \circ \mathcal{V} = \mathcal{V} \circ \mathcal{U}$.

Of course for any set X, $\mathcal{D}$ and $\mathcal{I}$ permute with any quasi-uniformity $\mathcal{U}$ on X. Note that if a quasi-uniformity $\mathcal{U}$ permutes with a quasi-uniformity $\mathcal{V}$, then $\mathcal{U}^{-1}$ permutes with $\mathcal{V}^{-1}$, too. The following result should be compared with [25, Proposition 3.2] dealing with uniformities.

Lemma 1. *Let $\mathcal{U}$ and $\mathcal{V}$ be quasi-uniformities on a set X. Then the following conditions are equivalent*:

(a) *$\mathcal{U} \circ \mathcal{V}$ is a quasi-uniformity.*
(b) $\mathcal{U} \circ \mathcal{V} = \mathcal{U} \wedge \mathcal{V}$.
(c) $\mathcal{U} \circ \mathcal{V} \subseteq \mathcal{V} \circ \mathcal{U}$.

Proof. (a) → (b) We obviously have $\mathcal{U} \circ \mathcal{V} \subseteq \mathcal{U} \cap \mathcal{V}$. Therefore $\mathcal{U} \circ \mathcal{V} \subseteq \mathcal{U} \wedge \mathcal{V}$, if $\mathcal{U} \circ \mathcal{V}$ is a quasi-uniformity. On the other hand let $H \in \mathcal{U} \wedge \mathcal{V}$. There are $U \in \mathcal{U}$ and $V \in \mathcal{V}$ such that $(U \cup V)^2 \subseteq H$. Consequently $\mathcal{U} \wedge \mathcal{V} \subseteq \mathcal{U} \circ \mathcal{V}$. Therefore $\mathcal{U} \wedge \mathcal{V} = \mathcal{U} \circ \mathcal{V}$.

(b) → (c) Let $H \in \mathcal{U} \wedge \mathcal{V}$. Analogously as in the first part of this proof, there are $U \in \mathcal{U}$ and $V \in \mathcal{V}$ such that $(U \cup V)^2 \subseteq H$. Therefore $H \in \mathcal{V} \circ \mathcal{U}$. Consequently $\mathcal{U} \circ \mathcal{V} = \mathcal{U} \wedge \mathcal{V} \subseteq \mathcal{V} \circ \mathcal{U}$.

(c) → (a) Let $U \in \mathcal{U}$ and $V \in \mathcal{V}$ be given. Then there are $U' \in \mathcal{U}$ and $V' \in \mathcal{V}$ such that $U'^2 \subseteq U$ and $V'^2 \subseteq V$. Furthermore there are $U'' \in \mathcal{U}$ and $V'' \in \mathcal{V}$ such that $V'' \subseteq V'$, $U'' \subseteq U'$ and $V'' \circ U'' \subseteq U' \circ V'$. Consequently $(U'' \circ V'')^2 \subseteq U'' \circ V'' \circ U'' \circ V'' \subseteq U'' \circ U' \circ V' \circ V'' \subseteq U'^2 \circ V'^2 \subseteq U \circ V$. Thus $\mathcal{U} \circ \mathcal{V}$ is a quasi-uniformity. □

Remark 1. Let $\mathcal{U}$ and $\mathcal{V}$ be quasi-uniformities on a set X. Then $\mathcal{U} \circ \mathcal{V} = \mathcal{U} \cap \mathcal{V}$ implies that $\mathcal{V} \circ \mathcal{U} \subseteq \mathcal{U} \circ \mathcal{V}$ (hence that $\mathcal{V} \circ \mathcal{U}$ is a quasi-uniformity by Lemma 1), since $\mathcal{V} \circ \mathcal{U} \subseteq \mathcal{U} \cap \mathcal{V}$.

Corollary 1. *(Compare [28, Proposition 3.1].) Two quasi-uniformities $\mathcal{U}$ and $\mathcal{V}$ permute if and only if both $\mathcal{U} \circ \mathcal{V}$ and $\mathcal{V} \circ \mathcal{U}$ are quasi-uniformities.*

Proof. This is an immediate consequence of Lemma 1. □

Corollary 2. *Let $\mathcal{U}$ and $\mathcal{V}$ be quasi-uniformities on a set X. If $\mathcal{U}\circ\mathcal{V}$ is a quasi-uniformity, then $\mathcal{V}^{-1}\circ\mathcal{U}^{-1}$ is a quasi-uniformity, too.*

Proof. If $\mathcal{U}\circ\mathcal{V}$ is a quasi-uniformity, then by Lemma 1, $\mathcal{U}\circ\mathcal{V}=\mathcal{U}\wedge\mathcal{V}$. Therefore $\mathcal{V}^{-1}\circ\mathcal{U}^{-1}=(\mathcal{U}\wedge\mathcal{V})^{-1}$ is a quasi-uniformity, indeed equal to $\mathcal{U}^{-1}\wedge\mathcal{V}^{-1}$, since the operations of conjugation and $\wedge$ commute (see [4, p. 3141]). □

Corollary 3. *Suppose that for quasi-uniformities $\mathcal{U}$ and $\mathcal{V}$ on a set X we have that $\mathcal{U}\cap\mathcal{V}=\mathcal{U}\wedge\mathcal{V}$. Then $\mathcal{U}$ and $\mathcal{V}$ permute.* (*In particular, if $\mathcal{U}$ and $\mathcal{V}$ are quasi-uniformities on X such that $\mathcal{U}\subseteq\mathcal{V}$, then $\mathcal{U}$ and $\mathcal{V}$ permute.*)

Proof. According to the proof of Lemma 1 for any quasi-uniformities $\mathcal{U}$ and $\mathcal{V}$ on X we have $\mathcal{U}\wedge\mathcal{V}\subseteq\mathcal{U}\circ\mathcal{V}$, $\mathcal{V}\circ\mathcal{U}\subseteq\mathcal{U}\cap\mathcal{V}$. Hence the assertion obviously follows from our hypothesis. □

Corollary 4. *Let $\mathcal{U}$ and $\mathcal{V}$ be permuting quasi-uniformities on a set X and let $\mathcal{A}$ and $\mathcal{B}$ be quasi-uniformities on X such that $\mathcal{U}\wedge\mathcal{V}\subseteq\mathcal{A}\subseteq\mathcal{U}$ and $\mathcal{U}\wedge\mathcal{V}\subseteq\mathcal{B}\subseteq\mathcal{V}$. Then $\mathcal{A}$ and $\mathcal{B}$ permute, too.*

Proof. We make repeatedly use of Lemma 1. Since $\mathcal{U}\circ\mathcal{V}$ is a quasi-uniformity, we see that $\mathcal{U}\wedge\mathcal{V}\subseteq\mathcal{A}\wedge\mathcal{B}\subseteq\mathcal{A}\circ\mathcal{B}\subseteq\mathcal{U}\circ\mathcal{V}=\mathcal{U}\wedge\mathcal{V}$ and thus $\mathcal{A}\circ\mathcal{B}$ is a quasi-uniformity. Similarly one deduces that $\mathcal{B}\circ\mathcal{A}$ is a quasi-uniformity. In particular $\mathcal{A}$ and $\mathcal{B}$ permute. □

Corollary 5. *Let $\mathcal{U}$ and $\mathcal{V}$ be quasi-uniformities on a set X. Then $\mathcal{U}$ and $\mathcal{V}$ are permutable and $\mathcal{U}\wedge\mathcal{V}=\mathcal{I}$ if and only if $U\circ V=V\circ U=X\times X$ whenever $U\in\mathcal{U}$ and $V\in\mathcal{V}$.*

Proof. The statement is obvious by Lemma 1. □

Following for instance Weber [25, Proposition 4.3], who considered the uniform case, we shall call quasi-uniformities $\mathcal{U}$ and $\mathcal{V}$ on a set X *independent* provided that the condition of Corollary 5 is satisfied.

Remark 2. Let $\mathcal{U}$ and $\mathcal{V}$ be uniformities on a set X. Then $\mathcal{U}$ and $\mathcal{V}$ permute if and only if $\mathcal{U}\circ\mathcal{V}$ is symmetric.

Proof. If $\mathcal{U}$ and $\mathcal{V}$ permute, then $\mathcal{U}\circ\mathcal{V}$ is a quasi-uniformity by Lemma 1 that is symmetric (that is, a uniformity), because $(\mathcal{U}\circ\mathcal{V})^{-1}=\mathcal{V}^{-1}\circ\mathcal{U}^{-1}=\mathcal{V}\circ\mathcal{U}=\mathcal{U}\circ\mathcal{V}$.

For the converse suppose that $\mathcal{U}\circ\mathcal{V}$ is symmetric. Then $\mathcal{U}\circ\mathcal{V}=(\mathcal{U}\circ\mathcal{V})^{-1}=\mathcal{V}^{-1}\circ\mathcal{U}^{-1}=\mathcal{V}\circ\mathcal{U}$. Hence $\mathcal{U}$ and $\mathcal{V}$ permute. □

We next give an example of two quasi-uniformities $\mathcal{U}$ and $\mathcal{V}$ on a set X such that $\mathcal{U}\circ\mathcal{V}$ is a quasi-uniformity, but $\mathcal{U}$ and $\mathcal{V}$ are not permutable (see also [28, Example 3.1]).

Example 1. Let $X=\{0,1,2\}$. Set $A=\{0\}$ and $B=\{0,1\}$. Then the set-theoretic complement of $S_A\cup S_B$ in $X\times X$ is $A\times(X\setminus B)=\{(0,2)\}$, because $A\subseteq B$. Since $(0,1)\in S_B$ and $(1,2)\in S_A$, we conclude that $S_A\circ S_B=X\times X$. On the other hand, clearly $(0,2)\notin S_B\circ S_A$. Thus $\mathcal{U}_{S_A}\circ\mathcal{U}_{S_B}=\mathcal{I}$, but the filter $\mathcal{U}_{S_B}\circ\mathcal{U}_{S_A}$ is strictly finer than $\mathcal{I}$ and is not a quasi-uniformity on X.

Example 1 can be generalized as follows:

Example 2. Let A, B be distinct proper nonempty subsets A and B of a set X.

(a) Then $B\setminus A\neq\emptyset$ if and only if $\mathcal{U}_{S_A}\circ\mathcal{U}_{S_B}=\mathcal{I}$.
(b) The two atoms $\mathcal{U}_{S_A}$ and $\mathcal{U}_{S_B}$ of $(q(X),\subseteq)$ permute if and only if $A\setminus B\neq\emptyset$ and $B\setminus A\neq\emptyset$.

Proof. Suppose that $B\setminus A\neq\emptyset$. Note that $S_A\cup S_B$ has the complement $(A\cap B)\times(X\setminus(A\cup B))$ in $X\times X$. If this latter set is empty, then $S_A\cup S_B=X\times X$, and thus $X\times X\subseteq(S_A\circ\Delta)\cup(\Delta\circ S_B)\subseteq S_A\circ S_B$. So let us consider the case that $(A\cap B)\times(X\setminus(A\cup B))\neq\emptyset$. By assumption we find $z\in A\setminus B$. Consider arbitrary $x\in A\cap B$ and $y\in X\setminus(A\cup B)$. Then $(x,z)\in S_A$ and $(z,y)\in S_B$ and therefore obviously $S_B\circ S_A=X\times X$ in this case, too.

We conclude that $\mathcal{U}_{S_B}\circ\mathcal{U}_{S_A}=\mathcal{I}$ provided that $B\setminus A\neq\emptyset$.

For the converse assume that $B\subseteq A$. We can choose $b\in B$ and $c\in X\setminus A$ and conclude that $(b,c)\notin S_A\circ S_B$. Hence $\mathcal{U}_{S_A}\circ\mathcal{U}_{S_B}\neq\mathcal{I}$.

(b) If $A\setminus B\neq\emptyset$ and $B\setminus A\neq\emptyset$, then by (a) we have $\mathcal{U}_{S_B}\circ\mathcal{U}_{S_A}=\mathcal{I}=\mathcal{U}_{S_A}\circ\mathcal{U}_{S_B}$ and conclude that $\mathcal{U}_{S_A}$ and $\mathcal{U}_{S_B}$ permute.

For the converse suppose that $\mathcal{U}_{S_A}$ and $\mathcal{U}_{S_B}$ permute. Since $\mathcal{I}=\mathcal{U}_{S_B}\wedge\mathcal{U}_{S_A}$, because $\mathcal{U}_{S_A}$ and $\mathcal{U}_{S_B}$ are distinct atoms, and since by Lemma 1 $\mathcal{U}_{S_A}\circ\mathcal{U}_{S_B}=\mathcal{U}_{S_B}\circ\mathcal{U}_{S_A}=\mathcal{U}_{S_A}\wedge\mathcal{U}_{S_B}$, we conclude by (a) that $A\setminus B\neq\emptyset$ and $B\setminus A\neq\emptyset$. □

Corollary 6. *Let A be a proper nonempty subset of a set X. Then $\mathcal{U}_{S_A}$ permutes with $(\mathcal{U}_{S_A})^{-1}$.*

Proof. The statement follows from Example 2, since $(\mathcal{U}_{S_A})^{-1} = \mathcal{U}_{S_{X\setminus A}}$. $\square$

In connection with Corollary 6 let us remark that Section 5 of the present paper is devoted to quasi-uniformities that permute with their conjugate.

Example 3. (Compare e.g. [18, Proposition 6] for the uniform case.) Let $(X, \mathcal{U}, \leqslant)$ be a quasi-uniform space $(X, \mathcal{U})$ equipped with a lattice order $\leqslant$ such that the lattice operations $\wedge : (X \times X, \mathcal{U} \times \mathcal{U}) \to (X, \mathcal{U})$ and $\vee : (X \times X, \mathcal{U} \times \mathcal{U}) \to (X, \mathcal{U})$ are uniformly continuous. Then $\mathcal{U}_{\leqslant}$ and $\mathcal{U}$ permute.

Proof. Let $V \in \mathcal{U}$. By our assumption there is $W \in \mathcal{U}$ such that $(x', x'') \in W$ and $(y', y'') \in W$ imply that $(x' \wedge y', x'' \wedge y'') \in V$.

We show that $(W \circ \leqslant) \subseteq (\leqslant \circ V)$. Indeed let $x \leqslant t$ and $(t, y) \in W$. Then $(x, x) \in W$ and $(t, y) \in W$, which implies that $(x \wedge t, x \wedge y) \in V$. Consequently $(x, x \wedge y) \in V$ and $x \wedge y \leqslant y$. We conclude that $\mathcal{U}_{\leqslant} \circ \mathcal{U} \subseteq \mathcal{U} \circ \mathcal{U}_{\leqslant}$.

Let $V \in \mathcal{U}$. By our assumption there is $W \in \mathcal{U}$ such that $(x', x'') \in W$ and $(y', y'') \in W$ imply that $(x' \vee y', x'' \vee y'') \in V$.

Similarly as above we show that $(\leqslant \circ W) \subseteq (V \circ \leqslant)$. Let $(x, t) \in W$ and $t \leqslant y$. Then $(x, t) \in W$ and $(y, y) \in W$, which implies that $(x \vee y, t \vee y) \in V$. Thus $x \leqslant x \vee y$ and $(x \vee y, y) \in V$. We conclude that $\mathcal{U} \circ \mathcal{U}_{\leqslant} \subseteq \mathcal{U}_{\leqslant} \circ \mathcal{U}$. We have shown that $\mathcal{U}$ and $\mathcal{U}_{\leqslant}$ permute, as asserted. $\square$

4. Permutable complements

This section is motivated by a question posed in [8, Problem 1]. Suppose that $\mathcal{U}$ and $\mathcal{V}$ are two complementary quasi-uniformities on a set X. Is there always a maximal complement $\mathcal{V}'$ of $\mathcal{U}$ finer than $\mathcal{V}$ in $(q(X), \subseteq)$? In [8, Corollary 6] it was shown that a kind of transitive variant of this question has a positive answer. Below we indicate another variant of this problem that has a positive solution.

Definition 2. Let $\mathcal{U}$ be a quasi-uniformity on a set X. Then $\mathcal{V}$ will be called a *permutable complement* of $\mathcal{U}$ if $\mathcal{V}$ is a complement of $\mathcal{U}$ in $(q(X), \subseteq)$, and furthermore $\mathcal{V}$ and $\mathcal{U}$ permute. (Hence $\mathcal{U}$ and $\mathcal{V}$ are independent.)

For any subset A of a set X we have $\mathcal{U}_{S_A} \cap \mathcal{U}_{C_A} = \mathcal{I}$, as we noted in Section 2. Hence $\mathcal{U}_{S_A}$ and $\mathcal{U}_{C_A}$ are permutable complements in $(q(X), \subseteq)$ by Corollary 3.

Example 4. Let $X = \{0, 1, 2\}$. Furthermore let $A = \{0\}$ and $T = \Delta \cup \{(0, 1)\}$. Then $\mathcal{U}_{S_A}$ and $\mathcal{U}_T$ are complements in $(q(X), \subseteq)$ by [8, Remark 1], but since $(0, 2) \notin T \circ S_A$, we see that $\mathcal{U}_{S_A}$ and $\mathcal{U}_T$ do not permute (compare Corollary 5).

The following result and its proof should be compared with [25, Proposition 3.7].

Proposition 1.

(a) *Let $(\mathcal{U}_i)_{i\in I}$ be a chain of quasi-uniformities on a set X. Suppose that $\mathcal{U} \circ \mathcal{U}_i$ is a quasi-uniformity whenever $i \in I$. Then $\mathcal{U} \circ \bigvee_{i\in I} \mathcal{U}_i$ is a quasi-uniformity and $\bigvee_{i\in I} (\mathcal{U} \wedge \mathcal{U}_i) = \mathcal{U} \circ (\bigvee_{i\in I} \mathcal{U}_i) = \mathcal{U} \wedge (\bigvee_{i\in I} \mathcal{U}_i)$.*

(b) *Let $(\mathcal{U}_i)_{i\in I}$ be a chain of quasi-uniformities on a set X where each $\mathcal{U}_i$ permutes with some given quasi-uniformity $\mathcal{U}$ on X. Then $\mathcal{U}$ and $\bigvee_{i\in I} \mathcal{U}_i$ permute.*

Proof. (a) By Lemma 1 and the fact that the family $(\mathcal{U}_i)_{i\in I}$ is a chain, the quasi-uniformity $\bigvee_{i\in I} (\mathcal{U} \wedge \mathcal{U}_i)$ has a base of the form $\{U \circ U_i\colon\ U \in \mathcal{U},\ U_i \in \mathcal{U}_i,\ i \in I\}$. This base also generates the filter $\mathcal{U} \circ (\bigvee_{i\in I} \mathcal{U}_i)$ on $X \times X$, which therefore is a quasi-uniformity and, thus, by Lemma 1, coincides with $\mathcal{U} \wedge (\bigvee_{i\in I} \mathcal{U}_i)$. Hence the equalities are established.

(b) Since $\mathcal{U}_i \circ \mathcal{U} = \mathcal{U} \circ \mathcal{U}_i$ whenever $i \in I$, by an argument similar to that presented in part (a) we have $\mathcal{U} \circ (\bigvee_{i\in I} \mathcal{U}_i) = \bigvee_{i\in I} (\mathcal{U} \circ \mathcal{U}_i) = \bigvee_{i\in I} (\mathcal{U}_i \circ \mathcal{U}) = (\bigvee_{i\in I} \mathcal{U}_i) \circ \mathcal{U}$. $\square$

Proposition 2. *Let $\mathcal{U}$ be a quasi-uniformity on a set X and let $\mathcal{V}$ be a permutable complement of $\mathcal{U}$. Then there is a maximal permutable complement $\mathcal{V}'$ of $\mathcal{U}$ finer than $\mathcal{V}$ in $(q(X), \subseteq)$.*

Proof. By Zorn's Lemma consider a maximal chain $(\mathcal{V}_i)_{i\in I}$ of permutable complements of $\mathcal{U}$ finer than $\mathcal{V}$. Then by Proposition 1 (and Lemma 1) $\bigvee_{i\in I} \mathcal{V}_i$ is a maximal permutable complement of $\mathcal{U}$ finer than $\mathcal{V}$, since $(\bigvee_{i\in I} \mathcal{V}_i) \wedge \mathcal{U} = \bigvee_{i\in I} (\mathcal{V}_i \wedge \mathcal{U}) = \mathcal{I}$. $\square$

5. Quasi-uniformities that permute with their conjugate

It is natural to ask which quasi-uniformities permute with their conjugate. Below we present a complete characterization of those Pervin quasi-uniformities of topological spaces that have this property. But the general, apparently difficult problem remains unresolved. Of course, trivially, each uniformity has the considered property. Also, for instance, for each linear order $\leqslant$ on a set X, obviously $\mathcal{U}_{\leqslant}$ and $(\mathcal{U}_{\leqslant})^{-1}$ are permutable complements in $(q(X), \subseteq)$ by Corollary 3.

Among other things the following result can be applied to proximally nondiscrete anti-atoms of $(q(X), \subseteq)$ (see [4, Theorem 1]).

Example 5. Let A be a proper nonempty subset of a set X. Suppose that $\mathcal{G}$ is a quasi-uniformity on X such that $C_A \in \mathcal{G}$.

Then $\mathcal{G} \cap \mathcal{G}^{-1} = \mathcal{G} \circ \mathcal{G}^{-1}$ if and only if there is $G \in \mathcal{G}$ such that $|G(x)| \leqslant 2$ whenever $x \in X$. (Note that the latter condition implies that $\mathcal{G}^{-1} \circ \mathcal{G}$ is a quasi-uniformity by Remark 1 and Lemma 1.)

Proof. Suppose that $\mathcal{G}$ satisfies the given property. Let $H \in \mathcal{G}$ be such that $H \subseteq G$. Then $H \circ H^{-1} = H \cup H^{-1}$: Suppose otherwise. Then there is $(x, y) \in H \circ H^{-1} \setminus (H \cup H^{-1})$. Therefore there is $a \in X$ such that $(a, x) \in H$ and $(a, y) \in H$. Note that $x \neq a$ and $y \neq a$, because $(x, y) \notin H$ and $(y, x) \notin H$.

By our assumption that $|G(a)| \leqslant 2$, we get that $|H(a)| \leqslant 2$ and conclude that $x = y$, which contradicts $(x, y) \notin H \cup H^{-1}$.

It follows that indeed $H \circ H^{-1} = H \cup H^{-1}$ and thus $\mathcal{G} \circ \mathcal{G}^{-1} = \mathcal{G} \cap \mathcal{G}^{-1}$.

In order to prove the converse, suppose that $\mathcal{G} \circ \mathcal{G}^{-1} = \mathcal{G} \cap \mathcal{G}^{-1}$. Therefore $\mathcal{G}^{-1} \circ \mathcal{G} \subseteq \mathcal{G} \circ \mathcal{G}^{-1}$ by Remark 1. There is $G \in \mathcal{G}$ such that $G \subseteq C_A$ by our assumption. Furthermore there is $H \in \mathcal{G}$ such that $H \subseteq G$ and $H \circ H^{-1} \subseteq G^{-1} \circ G$.

Suppose that $a \in X$ and $x, y \in H(a)$ such that $x \neq a$ and $y \neq a$. It follows that $a \in A$ and $x, y \in X \setminus A$, since $H(a) \subseteq C_A(a)$ and x and y are distinct from a.

Furthermore $(x, y) \in H \circ H^{-1} \subseteq G^{-1} \circ G$. Then there is $b \in X$ such that $(x, b) \in G$ and $(y, b) \in G$. Note that $b = x$ and $b = y$, since $x \in X \setminus A$, $y \in X \setminus A$ and $G \subseteq C_A$.

Hence $x = y$. We conclude that $|H(a)| \leqslant 2$ whenever $a \in X$. Hence the stated condition for $\mathcal{G}$ is satisfied. □

Corollary 7. *Let $\mathcal{G}$ be a quasi-uniformity on a set X such that for some proper nonempty subset A of X we have that $C_A \in \mathcal{G}$. Then $\mathcal{G}$ and $\mathcal{G}^{-1}$ permute if and only if there is $G \in \mathcal{G}$ such that $|G(x)| \leqslant 2$ and $|G^{-1}(x)| \leqslant 2$ whenever $x \in X$.*

Proof. Note first that $(C_A)^{-1} = C_{X \setminus A}$. If the stated condition is satisfied, then $\mathcal{G}$ and $\mathcal{G}^{-1}$ permute, since by applying Example 5 to $\mathcal{G}$ and $\mathcal{G}^{-1}$, we conclude that $\mathcal{G}^{-1} \circ \mathcal{G} = \mathcal{G} \cap \mathcal{G}^{-1} = \mathcal{G} \circ \mathcal{G}^{-1}$.

In order to prove the converse suppose that $\mathcal{G}$ and $\mathcal{G}^{-1}$ permute. Then $\mathcal{G}^{-1} \circ \mathcal{G} \subseteq \mathcal{G} \circ \mathcal{G}^{-1}$ and $\mathcal{G} \circ \mathcal{G}^{-1} \subseteq \mathcal{G}^{-1} \circ \mathcal{G}$. By applying the second part of the proof of Example 5 to $\mathcal{G}$ and $\mathcal{G}^{-1}$ we get that there are $G_1 \in \mathcal{G}$ and $G_2 \in \mathcal{G}$ such that $|G_1(x)| \leqslant 2$ and $|G_2^{-1}(x)| \leqslant 2$ whenever $x \in X$. Then we can set $G := G_1 \cap G_2$, and G fulfills the stated condition. □

Example 6. This example illustrates Example 5.

(a) Let $\mathcal{E}$ be a free ultrafilter on a countably infinite set X and let $x \in X$. Let $\mathcal{G}$ be the quasi-uniformity on X generated by $\{\Delta \cup (\{x\} \times E)\colon\ E \in \mathcal{E}\}$. Note that $C_{\{x\}} \in \mathcal{G}$. Observe also that $\mathcal{G}^{-1} \circ \mathcal{G} = \mathcal{G} \cap \mathcal{G}^{-1}$, which is not a quasi-uniformity on X. On the other hand $\mathcal{G} \circ \mathcal{G}^{-1}$ is the uniformity generated by the base $\{\Delta \cup (E \times E)\colon\ E \in \mathcal{E}\}$.

(b) Using the notation of part (a), assume that $y \in X$ and $y \neq x$. Let $\mathcal{H} = \mathcal{U}_T$ where $T = \Delta \cup \{(x, y)\}$. Then $\mathcal{H}$ and $\mathcal{H}^{-1}$ permute.

Let us recall that a topological space is called *normal* [9, p. 40] provided that any two disjoint closed sets can be separated by open sets. A topological space is called *extremally disconnected* (compare [9, p. 368]) if any two disjoint open sets have disjoint closures. Hence in some sense the properties of normality and extremal disconnectedness are dual to each other (see for instance [14, p. 301]).[2]

We shall use below the fact that a topological space X is extremally disconnected if and only if for any two open sets G_1 and G_2 of X we have $\overline{G_1 \cap G_2} = \overline{G_1} \cap \overline{G_2}$ [12, Theorem 3]. Let us sketch an argument for the nontrivial part of this statement: Indeed suppose that there is $x \in (\overline{G_1} \cap \overline{G_2}) \setminus \overline{G_1 \cap G_2}$ and let N be an open neighborhood of x contained in $X \setminus \overline{G_1 \cap G_2}$. Then $x \in \overline{G_1 \cap N}$ and $x \in \overline{G_2 \cap N}$, but $(G_1 \cap N) \cap (G_2 \cap N) = G_1 \cap G_2 \cap N = \emptyset$. We have reached a contradiction and conclude that the statement holds. By induction it follows that for any nonempty finite open collection $\mathcal{M}$ of an extremally disconnected space we have that $\overline{\bigcap_{M \in \mathcal{M}} M} = \bigcap_{M \in \mathcal{M}} \overline{M}$.

Let us also recall that a topological space is extremally disconnected if and only each open subset has an open closure (see [9, Theorem 6.2.26]).

[2] Contrary to the usage in Ref. [9] we shall not assume that normal or extremally disconnected spaces are Hausdorff spaces by definition. Instead we shall explicitly state such (additional) separation assumptions.

As usual, given an interior-preserving open collection $\mathcal{C}$ in a topological space X we shall define the transitive neighbornet (= neighborhood assignment) $T_{\mathcal{C}} = \bigcap_{C \in \mathcal{C}} S_C$ of X in the sense of the so-called Fletcher construction (compare [11, p. 29]).

Because of the aforementioned duality between the two properties of normality and extremal disconnectedness, in the following several arguments related to these properties look necessarily somewhat similar, but for the convenience of the reader we shall include both variants of proofs.

Lemma 2. *Let $\mathcal{W}$ be a compatible quasi-uniformity on a topological space X that is finer than the Pervin quasi-uniformity of X.*

(a) *Then X is normal if $\mathcal{W} \circ \mathcal{W}^{-1}$ is a (quasi-)uniformity.*
(b) *Then X is extremally disconnected if $\mathcal{W}^{-1} \circ \mathcal{W}$ is a (quasi-)uniformity.*

Proof. (a) Suppose that $\mathcal{W} \circ \mathcal{W}^{-1}$ is a (quasi-)uniformity, which is then indeed equal to the uniformity $\mathcal{W} \wedge \mathcal{W}^{-1}$ according to Lemma 1. Let F_1 and F_2 be two arbitrary disjoint closed sets in X. Set $P = S_{X \setminus F_1} \cap S_{X \setminus F_2}$. Then $P \in \mathcal{W}$. Note that $P^{-1} = S_{F_1} \cap S_{F_2}$ and $P \circ P^{-1} = \bigcup_{x \in X} (P(x) \times P(x)) = (X \setminus F_1)^2 \cup (X \setminus F_2)^2$. Now recall that $P \circ P^{-1} \in \mathcal{W} \circ \mathcal{W}^{-1} = \mathcal{W} \wedge \mathcal{W}^{-1}$, which is a (quasi-)uniformity. Thus there is $W \in \mathcal{W} \wedge \mathcal{W}^{-1}$ such that $W^4 \subseteq P \circ P^{-1}$. Then there is $Q \in \mathcal{W}$ such that $Q \cup Q^{-1} \subseteq W$. It follows that $(\bigcup_{x \in X} (Q^{-1}(x) \times Q^{-1}(x)))^2 \subseteq (Q^{-1} \circ Q)^2 \subseteq P \circ P^{-1}$.

Assume that there is an $x \in X$ such that $x \in Q(F_1) \cap Q(F_2)$. Then we find $f_1 \in Q^{-1}(x) \cap F_1$ and $f_2 \in Q^{-1}(x) \cap F_2$. Therefore $(f_1, x) \in Q^{-1}(x) \times Q^{-1}(x)$ and $(x, f_2) \in Q^{-1}(x) \times Q^{-1}(x)$. Consequently $(f_1, f_2) \in (X \setminus F_1)^2 \cup (X \setminus F_2)^2$ which obviously cannot hold. We conclude that $Q(F_1) \cap Q(F_2) = \emptyset$, which implies that X is normal, since $\operatorname{int} Q(F_1)$ and $\operatorname{int} Q(F_2)$ are open sets containing F_1 resp. F_2.

(b) We suppose that $\mathcal{W}^{-1} \circ \mathcal{W}$ is a (quasi-)uniformity, which then equals $\mathcal{W} \wedge \mathcal{W}^{-1}$ according to Lemma 1. Let G_1 and G_2 be two arbitrary disjoint open sets in X. Set $P = S_{G_1} \cap S_{G_2}$. Then $P \in \mathcal{W}$. Note that $P^{-1} = S_{X \setminus G_1} \cap S_{X \setminus G_2}$ and $P^{-1} \circ P = \bigcup_{x \in X} (P^{-1}(x) \times P^{-1}(x)) = (X \setminus G_1)^2 \cup (X \setminus G_2)^2$. Observe that $P^{-1} \circ P \in \mathcal{W}^{-1} \circ \mathcal{W}$, which by our assumption is equal to the quasi-uniformity $\mathcal{W} \wedge \mathcal{W}^{-1}$. Thus there is $W \in \mathcal{W} \wedge \mathcal{W}^{-1}$ such that $W^4 \subseteq P^{-1} \circ P$. Therefore there is $Q \in \mathcal{W}$ such that $Q \cup Q^{-1} \subseteq W$. Consequently $(Q \circ Q^{-1})^2 \subseteq P^{-1} \circ P$. Suppose that there is an $x \in X$ such that $x \in Q^{-1}(G_1) \cap Q^{-1}(G_2)$. Then we can find $g_1 \in Q(x) \cap G_1$ and $g_2 \in Q(x) \cap G_2$. Therefore $(g_1, x) \in Q(x) \times Q(x) \subseteq Q \circ Q^{-1}$ and $(x, g_2) \in Q(x) \times Q(x) \subseteq Q \circ Q^{-1}$. Thus $(g_1, g_2) \in C \times C$ where $C = X \setminus G_1$ or $C = X \setminus G_2$, which obviously cannot hold. It follows that $Q^{-1}(G_1) \cap Q^{-1}(G_2) = \emptyset$, which implies that the closures of G_1 and G_2 are disjoint in X. Hence X is extremally disconnected. $\square$

By definition, for any topological space X, the base $\{T_{\mathcal{L}}\colon \mathcal{L}$ is a locally finite open cover of $X\}$ generates the *locally finite open covering quasi-uniformity* of X (see [11, p. 30]).

Lemma 3. *Let X be a topological space and let $\mathcal{Q}$ be equal to the Pervin quasi-uniformity or the locally finite covering quasi-uniformity of X.*

(a) *If X is normal, then $\mathcal{Q} \circ \mathcal{Q}^{-1}$ is a (quasi-)uniformity.*
(b) *If X is extremally disconnected, then $\mathcal{Q}^{-1} \circ \mathcal{Q}$ is a (quasi-)uniformity.*

Proof. (a) Suppose that X is normal. Let $P \in \mathcal{Q}$. There is a (locally) finite open cover $\mathcal{C}$ of X such that $T_{\mathcal{C}} \subseteq P$ (compare [11, Proposition 2.7]). As it is well known, by normality of X there is a (locally) finite open cover $\mathcal{H}$ of X such that $\{\operatorname{st}(x, \mathcal{H})\colon x \in X\}$ refines the (locally) finite open cover $\mathcal{D} = \{T_{\mathcal{C}}(x)\colon x \in X\}$. (As usual, $\operatorname{st}(x, \mathcal{H}) = \bigcup\{H\colon x \in H \in \mathcal{H}\}$ whenever $x \in X$.) Since this argument is crucial, let us sketch it here.

Indeed, by normality (see [9, Theorem 1.5.18]), for each $D \in \mathcal{D}$ there is an open C_D such that $\overline{C_D} \subseteq D$ and $\{C_D\colon D \in \mathcal{D}\}$ is a cover of X. For each finite $\mathcal{D}' \subseteq \mathcal{D}$ set $H_{\mathcal{D}'} = \bigcap\{D\colon D \in \mathcal{D}'\} \setminus \bigcup\{\overline{C_D}\colon D \in \mathcal{D} \setminus \mathcal{D}'\}$ (see [9, Lemma 5.1.13 and Remark 5.1.14]). Set $\mathcal{H} = \{H_{\mathcal{D}'}\colon \mathcal{D}' \subseteq \mathcal{D},\ \mathcal{D}'$ is finite$\}$. Let $x \in X$ and $\mathcal{D}'_x = \{D \in \mathcal{D}\colon x \in \overline{C_D}\}$. Note that $x \in H_{\mathcal{D}'_x}$. Hence $\mathcal{H}$ is a (locally) finite open cover of X.

Fix any $D \in \mathcal{D}'_x$. Then $x \in \overline{C_D}$. If for some finite $\mathcal{D}' \subseteq \mathcal{D}$ we have $x \in H_{\mathcal{D}'}$, then $D \in \mathcal{D}'$ and thus $H_{\mathcal{D}'} \subseteq D$. Therefore $\{\operatorname{st}(x, \mathcal{H})\colon x \in X\}$ refines $\mathcal{D}$, which establishes the claim.

Of course, $T_{\mathcal{H}} \in \mathcal{Q}$. Moreover we have $(T_{\mathcal{H}} \circ T_{\mathcal{H}}^{-1})^2 = (T_{\mathcal{H}} \circ T_{\mathcal{H}}^{-1}) \circ (T_{\mathcal{H}} \circ T_{\mathcal{H}}^{-1})^{-1} = \bigcup_{x \in X} ((T_{\mathcal{H}} \circ T_{\mathcal{H}}^{-1})(x) \times (T_{\mathcal{H}} \circ T_{\mathcal{H}}^{-1})(x)) \subseteq \bigcup_{D \in \mathcal{D}} D \times D \subseteq T_{\mathcal{C}} \circ T_{\mathcal{C}}^{-1} \subseteq P \circ P^{-1}$. We conclude that $\mathcal{Q} \circ \mathcal{Q}^{-1}$ is a uniformity.

(b) Suppose that X is extremally disconnected. Let $P \in \mathcal{Q}$. Let $\mathcal{L}$ be a (locally) finite open cover of X such that $T_{\mathcal{L}} \subseteq P$. Then $\overline{\mathcal{L}} = \{\bar{L}\colon L \in \mathcal{L}\}$ is a (locally) finite open collection of X, since X is extremally disconnected. Let $S = T_{\overline{\mathcal{L}}} \cap T_{\overline{\mathcal{L}}}^{-1}$. Observe that for each $x \in X$, we have

$$S(x) = \bigcap\{\bar{L}\colon L \in \mathcal{L},\ x \in \bar{L}\} \setminus \bigcup\{\bar{L}\colon L \in \mathcal{L},\ x \notin \bar{L}\}.$$

Note that $\mathcal{S} = \{S(x)\colon x \in X\}$ is a (locally) finite open partition of X. So $T_{\mathcal{S}}$ belongs to the quasi-uniformity $\mathcal{Q}$ of X. Furthermore for each $x \in X$, $T_{\mathcal{S}}(x) = S(x)$.

Observe that for each $x \in X$, we have $T_{\mathcal{L}}^{-1}(x) = X \setminus \bigcup\{L:\ L \in \mathcal{L},\ x \notin L\}$ and $\operatorname{int} T_{\mathcal{L}}^{-1}(x) = X \setminus \bigcup\{\bar{L}:\ L \in \mathcal{L},\ x \notin L\}$, because $\mathcal{L}$ is locally finite. So if for $x, y \in X$ we have $x \in \operatorname{int} T_{\mathcal{L}}^{-1}(y)$, then $x \notin \bigcup\{\bar{L}:\ L \in \mathcal{L},\ y \notin L\}$; therefore $X \setminus \bigcup\{\bar{L}:\ L \in \mathcal{L},\ x \notin \bar{L}\} \subseteq X \setminus \bigcup\{\bar{L}:\ L \in \mathcal{L},\ y \notin L\}$ which means that $T_{\bar{\mathcal{L}}}^{-1}(x) \subseteq \operatorname{int} T_{\mathcal{L}}^{-1}(y)$. Consequently we only need to show that $\{\operatorname{int} T_{\mathcal{L}}^{-1}(y):\ y \in X\}$ is a cover in order to see that the partition $\{S(x):\ x \in X\}$ refines the cover $\{\operatorname{int} T_{\mathcal{L}}^{-1}(y):\ y \in X\}$. But this is a consequence of the following argument: We have that $\bigcap_{y \in X}(X \setminus T_{\mathcal{L}}^{-1}(y)) = \emptyset$, since $\{T_{\mathcal{L}}^{-1}(y):\ y \in X\}$ is a cover of X. Hence by the definition of $T_{\mathcal{L}}^{-1}(y)$ $(y \in X)$ we obtain $\bigcap_{y \in X}(\bigcup\{L \in \mathcal{L}:\ y \notin L\}) = \emptyset$.

In order to reach a contradiction suppose that there is

$$x \in \bigcap_{y \in X} \overline{\bigcup\{L:\ L \in \mathcal{L},\ y \notin L\}}.$$

Then $x \in \bigcap_{y \in X} \bigcup\{\bar{L}:\ L \in \mathcal{L},\ y \notin L\}$, because $\mathcal{L}$ is locally finite.

Therefore for each $y \in X$ there is $L_y \in \mathcal{L}$ such $x \in \overline{L_y}$ and $y \notin L_y$. However $\{L_y:\ y \in X\}$ is indeed a finite collection, since $\overline{\mathcal{L}}$ is locally finite. Then using the property of nonempty finite collections of open sets in an extremally disconnected space stated above, we see that $x \in \bigcap_{y \in X} \overline{L_y} = \overline{\bigcap_{y \in X} L_y} \subseteq \overline{\bigcap_{y \in X}(\bigcup\{L \in \mathcal{L}:\ y \notin L\})}$. But $\bigcap_{y \in X}(\bigcup\{L \in \mathcal{L}:\ y \notin L\}) = \emptyset$, as we noted above. We have reached a contradiction and conclude that

$$\bigcap_{y \in X}\left(\overline{\bigcup\{L \in \mathcal{L}:\ y \notin L\}}\right) = \emptyset.$$

Hence $\{\operatorname{int} T_{\mathcal{L}}^{-1}(y):\ y \in X\}$ is indeed a cover of X.

Furthermore by symmetry and transitivity of S we see that $(S^{-1} \circ S)^2 \subseteq \bigcup_{x \in X}((S^{-1} \circ S)(x) \times (S^{-1} \circ S)(x)) = \bigcup_{x \in X}(S(x) \times S(x)) \subseteq \bigcup_{y \in X}(T_{\mathcal{L}}^{-1}(y) \times T_{\mathcal{L}}^{-1}(y)) = T_{\mathcal{L}}^{-1} \circ T_{\mathcal{L}} \subseteq P^{-1} \circ P$. Thus $\mathcal{Q}^{-1} \circ \mathcal{Q}$ is a uniformity. □

Our next results describe the uniformities found in Lemma 3. In connection with Proposition 3 we recall that normal T_1-spaces and extremally disconnected regular spaces are completely regular.

Proposition 3. *Let (X, τ) be a topological space and let τ' be the finest completely regular topology on X coarser than τ. Furthermore let $(\mathcal{C}')^*$ be the finest totally bounded compatible uniformity on the completely regular space (X, τ'). Let $\mathcal{P}$ (resp. $\mathcal{P}'$) be the Pervin quasi-uniformity of (X, τ) (resp. (X, τ')). Let $(\mathcal{LF})'$ be the locally finite covering quasi-uniformity of τ'. Moreover let ϕ' be the fine uniformity of τ' and let $\mathcal{F}$ be a compatible quasi-uniformity for τ that is finer than the locally finite covering quasi-uniformity $\mathcal{LF}$ of τ.*

(a) *Then $(\mathcal{C}')^* = \mathcal{P}' \wedge (\mathcal{P}')^{-1} = \mathcal{P} \wedge \mathcal{P}^{-1}$.*
(b) *Moreover $\phi' = (\mathcal{LF})' \wedge ((\mathcal{LF})')^{-1} = \mathcal{F} \wedge \mathcal{F}^{-1}$.*

Proof. (a) This result was obtained in [6, Lemma 3].

(b) We have $\phi' \subseteq (\mathcal{LF})'$ by [11, Proposition 5.3] and obviously $(\mathcal{LF})' \subseteq \mathcal{LF} \subseteq \mathcal{F}$. Thus $\phi' \subseteq (\mathcal{LF})' \wedge ((\mathcal{LF})')^{-1} \subseteq \mathcal{F} \wedge \mathcal{F}^{-1}$, since ϕ' is a uniformity. Then $\tau' = \tau(\phi') \subseteq \tau(\mathcal{F} \wedge \mathcal{F}^{-1}) \subseteq \tau$ and $\tau(\mathcal{F} \wedge \mathcal{F}^{-1})$ is completely regular. Thus the uniformity $\mathcal{F} \wedge \mathcal{F}^{-1}$ induces the topology τ' and hence $\phi' = \mathcal{F} \wedge \mathcal{F}^{-1}$, because ϕ' is the finest uniformity compatible with τ'. Furthermore obviously $\phi' = (\mathcal{LF})' \wedge ((\mathcal{LF})')^{-1}$, too. □

In the remaining paragraphs of this section we shall make use of the notation introduced in Proposition 3, sometimes without further comments.

Corollary 8.

(a) *Let X be a topological space such that $\mathcal{P} \circ \mathcal{P}^{-1}$ (resp. $\mathcal{P}^{-1} \circ \mathcal{P}$) is a uniformity. Then this uniformity is equal to $(\mathcal{C}')^*$.*
(b) *Let X be a topological space such that $\mathcal{F} \circ \mathcal{F}^{-1}$ (resp. $\mathcal{F}^{-1} \circ \mathcal{F}$) is a uniformity. Then this uniformity is equal to ϕ'.*

Proof. The statement is obvious by Lemma 1 and Proposition 3. □

Remark 3. Note that in Lemma 3 for a normal space X the uniformity $\mathcal{P} \circ \mathcal{P}^{-1}$ is not transitive in general: Consider the real unit interval I with the usual topology. Then for the Pervin quasi-uniformity $\mathcal{P}$ of I, we have $\mathcal{P} \circ \mathcal{P}^{-1} = \mathcal{P} \wedge \mathcal{P}^{-1}$, which is the usual (nontransitive) Euclidean uniformity $(\mathcal{C}^*)'$ on I [6, Example 2]. On the other hand for any extremally disconnected space X the uniformity $\mathcal{P}^{-1} \circ \mathcal{P}$ is transitive, as the proof of Lemma 3 shows. (Obviously a similar result holds for $\mathcal{LF}$ and ϕ', instead of $\mathcal{P}$ and $(\mathcal{C}^*)'$.)

Corollary 9. *Let $\mathcal{Q}$ be the Pervin or the locally finite covering quasi-uniformity of a topological space X.*

(a) *Then $\mathcal{Q} \circ \mathcal{Q}^{-1}$ is a uniformity if and only if X is normal.*
(b) *Then $\mathcal{Q}^{-1} \circ \mathcal{Q}$ is a uniformity if and only if X is extremally disconnected.*
(c) *Then $\mathcal{Q}$ and $\mathcal{Q}^{-1}$ permute if and only if X is normal and extremally disconnected.*

Proof. (a) The assertion follows from Lemmas 2 and 3.

(b) The assertion follows from Lemmas 2 and 3.

(c) If X is normal and extremally disconnected, then by Lemmas 1 and 3 $\mathcal{Q} \circ \mathcal{Q}^{-1} = \mathcal{Q} \wedge \mathcal{Q}^{-1} = \mathcal{Q}^{-1} \circ \mathcal{Q}$, hence $\mathcal{Q}$ and $\mathcal{Q}^{-1}$ permute. On the other hand, if $\mathcal{Q}$ and $\mathcal{Q}^{-1}$ permute, then by Lemma 1 both $\mathcal{Q} \circ \mathcal{Q}^{-1}$ and $\mathcal{Q}^{-1} \circ \mathcal{Q}$ are uniformities and so by Lemma 2 X is extremally disconnected and normal. □

Example 7. Let $X = \{0, 1\}$ and let $\tau = \{\emptyset, X, \{1\}\}$ be the Sierpinski topology on X. Then X is normal and extremally disconnected. The Pervin quasi-uniformity of this space is $\mathcal{U}_{S_{\{1\}}}$. Clearly $\mathcal{P} \circ \mathcal{P}^{-1} = \mathcal{P}^{-1} \circ \mathcal{P} = \mathcal{I}$ (compare Corollary 6).

A topological space X is called *orthocompact* [11, p. 100] provided that for each open cover $\mathcal{C}$ of X there is a preorder T on X such that $\{T(x)\colon x \in X\}$ is an open refinement of $\mathcal{C}$.

Recall that a topological space X is called *almost* 2*-fully normal* (see e.g. [20, Theorem 2.6]) provided that the set $\mathcal{N}$ of all neighborhoods in $X \times X$ of the diagonal Δ of X forms a uniformity on the set X. The latter class of spaces is known under various other names, such as *divisible spaces* or *functionally* Δ*-paracompact spaces* (see [3, p. 2262]). Some further names of this property were listed in [17, p. 173]. It is known that each almost 2-fully normal space is collectionwise normal (compare [20, Theorem 2.9]). For a definition of the latter concept see e.g. [9, p. 305].

We next investigate how almost 2-full normality (as a strong normality property) is related to the problem under consideration.

Proposition 4. *Let X be an almost* 2*-fully normal space and let $\mathcal{F}$ be a compatible quasi-uniformity on X that is finer than its locally finite covering quasi-uniformity. Then $\mathcal{F} \circ \mathcal{F}^{-1}$ is a uniformity.*

Proof. Let $U \in \mathcal{F} \circ \mathcal{F}^{-1}$. Then there is $F \in \mathcal{F}$ such that $F \circ F^{-1} \subseteq U$ where $F \circ F^{-1} = \bigcup_{x \in X}(F(x) \times F(x))$. Therefore U is a neighborhood of the diagonal of X. Hence by Proposition 3 and the proof of Lemma 1 we have $\phi' = \mathcal{F} \wedge \mathcal{F}^{-1} \subseteq \mathcal{F} \circ \mathcal{F}^{-1} \subseteq \mathcal{N}$. In particular $\tau(\phi') \subseteq \tau(\mathcal{N})$.

On the other hand, by our assumption on X, since the uniformity $\mathcal{N}$ of $\tau \times \tau$-neighborhoods of Δ induces a completely regular topology on X coarser than the topology of X, we see that $\tau(\phi') = \tau(\mathcal{N})$ and thus $\mathcal{N} \subseteq \phi'$.

We conclude that $\mathcal{F} \circ \mathcal{F}^{-1} = \phi'$ is a uniformity. □

The next example shows that the converse of Proposition 4 does not hold.

Example 8. The following space Y attributed to Bing was discussed by Cohen in [2]. Let $Y = \omega_1 \times (\omega_1 + 1)$, where in the topological product each point of $\omega_1 \times \omega_1$ has been made isolated. Let $\mathcal{F}$ be a compatible quasi-uniformity on Y that is finer than the locally finite covering quasi-uniformity of Y. We are going to prove that $\mathcal{F} \circ \mathcal{F}^{-1}$ is a uniformity, although the space Y is known not to be almost 2-fully normal (see [2]).

Let $U \in \mathcal{F}$. Choose $V \in \mathcal{F}$ such that $V^2 \subseteq U$ and $V(x)$ is open whenever $x \in Y$ (compare [11, p. 3]). We want to show that we can find a locally finite open cover $\mathcal{C}$ of Y such that $(\bigcup_{C \in \mathcal{C}} C \times C) \subseteq (\bigcup_{y \in Y} U(y) \times U(y))$.

For each $\alpha \in \omega_1 \setminus \{0\}$ we find $\beta_\alpha < \alpha$ and $\gamma_\alpha \in \omega_1$ such that $(\beta_\alpha, \alpha] \times [\gamma_\alpha, \omega_1] \subseteq V(\alpha, \omega_1)$. By the Pressing-Down Lemma (see e.g. [19, p. 153]) there are $\beta \in \omega_1$ and an uncountable set $A \subseteq \omega_1$ such that $\beta_\alpha < \beta < \alpha$ whenever $\alpha \in A$. Set $G = \bigcup_{\alpha \in A} V(\alpha, \omega_1)$. Furthermore choose a finite collection $\mathcal{E}$ of $\{V(\alpha, \omega_1)\colon \alpha \in \omega_1\}$ covering the compact subspace $[0, \beta] \times \{\omega_1\}$ of Y. Set $\mathcal{C} = \mathcal{E} \cup \{G\} \cup \{\{y\}\colon y \in Y \setminus (\bigcup \mathcal{E} \cup G)\}$. Note that $\mathcal{C}$ is a locally finite open cover of Y, since indeed each member of $\mathcal{C}$ hits only finitely many members of $\mathcal{C}$.

We finally show that $(\bigcup_{C \in \mathcal{C}} C \times C) \subseteq (\bigcup_{y \in Y} U(y) \times U(y))$. Obviously we only need to check that $G \times G \subseteq \bigcup_{y \in Y} U(y) \times U(y)$.

Let $(x, y) \in (\bigcup_{\alpha \in A} V(\alpha, \omega_1)) \times (\bigcup_{\alpha' \in A} V(\alpha', \omega_1))$. Then there are $\alpha, \alpha' \in A$ such that $x \in V(\alpha, \omega_1)$ and $y \in V(\alpha', \omega_1)$. Suppose that $\alpha' \leqslant \alpha$. (The case that $\alpha' > \alpha$ can be dealt with similarly.) Then by definition of A, $(\alpha', \omega_1) \in V(\alpha, \omega_1)$ and therefore $V(\alpha', \omega_1) \subseteq V^2(\alpha, \omega_1)$. Consequently $(x, y) \in U(\alpha, \omega_1) \times U(\alpha, \omega_1)$, which verifies our claim.

Since Y is normal, there is a locally finite open cover $\mathcal{L}$ of Y such that the cover $\{(T_{\mathcal{L}} \circ T_{\mathcal{L}}^{-1})(x)\colon x \in X\}$ refines $\mathcal{C}$ (see for instance the proof of Lemma 3). Hence $T_{\mathcal{L}} \in \mathcal{F}$ and $(T_{\mathcal{L}} \circ T_{\mathcal{L}}^{-1})^2 = \bigcup_{x \in X}((T_{\mathcal{L}} \circ T_{\mathcal{L}}^{-1})(x) \times (T_{\mathcal{L}} \circ T_{\mathcal{L}}^{-1})(x)) \subseteq \bigcup_{C \in \mathcal{C}} C \times C \subseteq U \circ U^{-1}$. We conclude that $\mathcal{F} \circ \mathcal{F}^{-1}$ is a uniformity.

Example 8 is known to be monotonically normal (compare [13, p. 803]). Let us note that each orthocompact monotonically normal space is almost 2-fully normal (see [15, Corollary 1] or [24, Theorem 6]).

Proposition 5. *Let (X,τ) be an orthocompact space and suppose that $\mathcal{Z}\circ\mathcal{Z}^{-1}$ is a uniformity, where $\mathcal{Z}$ denotes the fine quasi-uniformity of X. Then (X,τ) is almost 2-fully normal.*

Proof. Let U be a $\tau\times\tau$-neighborhood of the diagonal of X. Then there is an open cover $\mathcal{C}$ of (X,τ) such that $\bigcup_{C\in\mathcal{C}}(C\times C)\subseteq U$. By orthocompactness of X let T be a transitive neighbornet on (X,τ) such that $\{T(x)\colon x\in X\}$ refines $\mathcal{C}$. Then $T\circ T^{-1}\subseteq\bigcup_{C\in\mathcal{C}}(C\times C)$. Furthermore $T\circ T^{-1}\in\mathcal{Z}\circ\mathcal{Z}^{-1}=\phi'$ by Proposition 3. Thus there is $N\in\phi'$ such that $N^2\subseteq T\circ T^{-1}$. Then N is a $\tau'\times\tau'$-neighborhood of Δ and, thus a $\tau\times\tau$-neighborhood of Δ. We have verified that (X,τ) is almost 2-fully normal. □

Corollary 10. *Let X be an orthocompact space and let $\mathcal{Z}$ be its fine quasi-uniformity. Then $\mathcal{Z}\circ\mathcal{Z}^{-1}$ is a uniformity if and only if X is almost 2-fully normal.*

Proof. The assertion follows from Propositions 4 and 5. □

Example 9. Let G be a normal, metacompact space that is not collectionwise normal. (Such an example is due to Bing, see e.g. [9, Problem 5.5.3(c)].) Let $\mathcal{Z}$ denote the fine quasi-uniformity of G.

Hence by Proposition 5, G does not satisfy that $\mathcal{Z}\circ\mathcal{Z}^{-1}$ is a uniformity, since G is orthocompact, but not almost 2-fully normal.

Of course, by Lemma 3, $\mathcal{P}\circ\mathcal{P}^{-1}$ as well as $\mathcal{LF}\circ(\mathcal{LF})^{-1}$ (where $\mathcal{P}$ denotes the Pervin quasi-uniformity and $\mathcal{LF}$ denotes the locally finite covering quasi-uniformity of G) are uniformities, since G is normal.

Example 10. We continue our discussion of Example 6(a). Recall that X is a countably infinite set and $x\in X$. Furthermore $\mathcal{E}$ is a free ultrafilter on X. We considered the quasi-uniformity $\mathcal{G}$ on X generated by the base $\{\Delta\cup(\{x\}\times E)\colon E\in\mathcal{E}\}$.

One readily verifies that $(X,\tau(\mathcal{G}))$ is an almost 2-fully normal, extremally disconnected T_1-space. Obviously $\mathcal{G}$ is the fine quasi-uniformity of the topological space $(X,\tau(\mathcal{G}))$. Furthermore $\phi'=\mathcal{G}\circ\mathcal{G}^{-1}$ is the fine uniformity of $(X,\tau(\mathcal{G}))$, but $\mathcal{G}^{-1}\circ\mathcal{G}$ is not a quasi-uniformity on X (compare Proposition 4).

Problem 1. Characterize those (extremally disconnected) topological spaces X such that $\mathcal{Z}^{-1}\circ\mathcal{Z}$ is a uniformity, where $\mathcal{Z}$ denotes the fine quasi-uniformity of X (compare Lemma 2).

Problem 2. Characterize those topological spaces X for which $\mathcal{Z}$ and $\mathcal{Z}^{-1}$ permute, where $\mathcal{Z}$ is the fine quasi-uniformity of X.

References

[1] J. Cao, J. Rodríguez-López, On hyperspace topologies via distance functionals in quasi-metric spaces, Acta Math. Hungar. 112 (2006) 249–268.
[2] H.J. Cohen, Sur un problème de M. Dieudonné, C. R. Acad. Sci. Paris 234 (1952) 290–292.
[3] D.K. Burke, R.Z. Buzyakova, More on separation of a diagonal, Topology Appl. 157 (2010) 2261–2270.
[4] E.P. de Jager, H.-P.A. Künzi, Atoms, anti-atoms and complements in the lattice of quasi-uniformities, Topology Appl. 153 (2006) 3140–3156.
[5] E.P. de Jager, H.-P.A. Künzi, Infima of quasi-uniform anti-atoms, Topology Appl. 153 (2006) 3327–3337.
[6] E.P. de Jager, H.-P.A. Künzi, Infima and complements in the lattice of quasi-uniformities, Topology Appl. 154 (2007) 2117–2126.
[7] E.P. de Jager, H.-P.A. Künzi, The lattice of quasi-uniformities, in: E. Pearl (Ed.), Open Problems in Topology II, Elsevier, 2007, pp. 691–696.
[8] E.P. de Jager, H.-P.A. Künzi, More about complements of quasi-uniformities, Topology Appl., in press.
[9] R. Engelking, General Topology, Heldermann, Berlin, 1989.
[10] M.J. Ferreira, J. Picado, Functorial quasi-uniformities on frames, Appl. Categ. Structures 13 (2005) 281–303.
[11] P. Fletcher, W.F. Lindgren, Quasi-Uniform Spaces, Dekker, New York, 1982.
[12] Vishvajit V.S. Gautam, Some equivalent of extremally disconnected spaces, Indian J. Pure Appl. Math. 32 (2001) 343–344.
[13] K.P. Hart, Strong collectionwise normality and M.E. Rudin's Dowker space, Proc. Amer. Math. Soc. 83 (1981) 802–806.
[14] W. Kotzé, T. Kubiak, Insertion of a measurable function, J. Austral. Math. Soc. Ser. A 57 (1994) 295–304.
[15] H.-P.A. Künzi, Generalizations of preorthocompactness, Questions Answers Gen. Topology 4 (1986) 15–35.
[16] H.-P.A. Künzi, An introduction to quasi-uniform spaces, in: Contemp. Math., vol. 486, Amer. Math. Soc., 2009, pp. 239–304.
[17] H.-P.A. Künzi, P. Fletcher, Some questions related to almost 2-fully normal spaces, Rocky Mountain J. Math. 15 (1985) 173–183.
[18] H.-P.A. Künzi, Z. Mushaandja, Topological ordered C- (resp. I-)spaces and generalized metric spaces, Topology Appl. 156 (2009) 2914–2922.
[19] A. Lévy, Basic Set Theory, Springer, Berlin, 1979.
[20] M.J. Mansfield, Some generalizations of full normality, Trans. Amer. Math. Soc. 86 (1957) 489–505.
[21] J. Rodríguez-López, S. Romaguera, Closedness of bounded convex sets of asymmetric normed linear spaces and the Hausdorff quasi-metric, Bull. Belg. Math. Soc. Simon Stevin 13 (2006) 551–562.
[22] S. Romaguera, O. Valero, A quantitative computational model for complete partial metric spaces via formal balls, Math. Structures Comput. Sci. 19 (2009) 541–563.
[23] W.H. Rowan, Algebras with a compatible uniformity, Algebra Universalis 47 (2002) 13–43.
[24] M.E. Rudin, I.S. Stares, J.E. Vaughan, From countable compactness to absolute countable compactness, Proc. Amer. Math. Soc. 125 (1997) 927–934.
[25] H. Weber, On lattices of uniformities, Order 24 (2007) 249–276.
[26] H. Weber, Boolean algebras of lattice uniformities and decompositions of modular functions, Ric. Mat. 58 (2009) 15–32.
[27] C.H. Yan, Distributive laws for commuting equivalence relations, Discrete Math. 181 (1998) 295–298.
[28] C.H. Yan, Commuting quasi-order relations, Discrete Math. 183 (1998) 285–292.